Positive curvature property for sub-Laplace on nilpotent Lie group of rank two

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Abstract

In this note, we concentrate on the sub-Laplace on the nilpotent Lie group of rank two, which is the infinitesimal generator of the diffusion generated by n Brownian motions and their $\frac{n(n-1)}{2}$ Lévy area processes, which is the simple extension of the sub-Laplace on the Heisenberg group $\mathbb{H}$. In order to study contraction properties of the heat kernel, we show that, as in the cases of the Heisenberg group and the three Brownian motion model, the restriction of the sub-Laplace acting on radial functions (see Definition 3.5) satisfies a positive Ricci curvature condition (more precisely a $CD(0,\infty)$ inequality, see Theorem 4.5, whereas the operator itself does not satisfy any $CD(r,\infty)$ inequality. From this we may deduce some useful, sharp gradient bounds for the associated heat kernel. It can be seen a generalization of the paper [22].

Keywords: Γ_2 curvature, Heat kernel, Gradient estimates, Sub-Laplace, Nilpotent Lie groups.

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1 Introduction

The elliptic case

Let M be a complete Riemannian manifold of dimension n and let $\mathcal{L} := \Delta + \nabla h$, where Δ is the Laplace-Beltrami operator. For $t \geq 0$, denote by P_t the heat semigroup generated by $\mathcal{L}$ (that is formally $P_t = \exp(t\mathcal{L})$). For smooth enough function f, g, one defines (see [1])

$$\Gamma(f,g) = |\nabla f|^2 = \frac{1}{2}(\mathcal{L}fg - f\mathcal{L}g - g\mathcal{L}f),$$

$$\Gamma_2(f,f) = \frac{1}{2}\big(\mathcal{L}\Gamma(f,f) - 2\Gamma(f,\mathcal{L}f)\big) = |\nabla\nabla f|^2 + (Ric - \nabla\nabla h)(\nabla f, \nabla f).$$

We have the following well-known proposition, see Proposition 3.3 in [1].
Proposition A. For every real $\rho \in \mathbb{R}$, the following are equivalent

(i). $CD(\rho,\infty)$ holds. That is $\Gamma_2(f,f) \geq \rho\Gamma(f,f)$.

(ii). For $t \geq 0$, $\Gamma(P_t f, P_t f) \leq e^{-2\rho t} P_t(\Gamma(f,f))$.

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(iii). For $t \geq 0$, $\Gamma(P_t f, P_t f)^{\frac{1}{2}} \leq e^{-\rho t} P_t(\Gamma(f,f)^{\frac{1}{2}})$.

Moreover, in [8], Engoulatov obtained the following gradient estimates for the associated heat kernels $p(t,x,y)$ in Riemannian manifolds.

Theorem B. Let M be a complete Riemannian of dimension n with Ricci curvature bounded from below, $Ric(M) \geq -\rho$, $\rho \geq 0$.

(i). Suppose a non-collapsing condition is satisfies on M, namely, there exist $t_0 > 0$, and $\nu_0 > 0$, such that for any $x \in M$, the volume of the geodesic ball of radius t_0 centered at x is not too small, $Vol(B_x(t_0)) \geq \nu_0$. Then there exist two constants $C(\rho, n, \nu_0, t_0)$ and $\bar{C}(t_0) > 0$, such that

$$|\nabla \log p(t,x,y)| \leq C(\rho, n, \nu_0, t_0)\left(\frac{d(x,y)}{t} + \frac{1}{\sqrt{t}}\right),$$

uniformly on $(0, \bar{C}(t_0)] \times M \times M$, where $d(x,y)$ is the Riemannian distance between x and y.

(ii). Suppose that M has a diameter bounded by D, Then there exists a constant $C(\rho, n)$ such that

$$|\nabla \log p(t,x,y)| \leq C(\rho, n)\left(\frac{D}{t} + \frac{1}{\sqrt{t}} + \rho\sqrt{t}\right),$$

uniformly on $(0, \infty) \times M \times M$.

Recently, X. D. Li [18] has shown that the non-collapsing condition can be removed.

The hypoelliptic case

More recent, some focus has been set on some degenerate (hypoelliptic) situations, where the methods used for the elliptic case do not apply. Among the simplest examples of such situation is the Heisenberg group, denote p_t the heat kernel of Markov semigroup P_t at the origin o with respect to Lebesgue measures on $\mathbb{R}^3$, see [9, 13, 14] for the precise formulas. H. Q. Li obtain the sharp gradient estimate for the heat kernel p_t and the contraction property for the semigroup P_t, which generalizes and strengthens the result of Driver and Melcher, [7].

Theorem C. For any $g \in \mathbb{H}$, we have

$$|\nabla \log p_t|(g) \leq \frac{Cd(g)}{t}, \tag{1.1}$$

where $d(g)$ is the Carnot-Carathéodory distance between o and g. In addition, we have

$$\forall f \in C_0^\infty(\mathbb{H}),\ \forall t \geq 0,\ \Gamma(P_t f, P_t f)^{\frac{1}{2}} \leq C_1 P_t\big(\Gamma(f,f)^{\frac{1}{2}}\big). \tag{1.2}$$

(See also D. Bakry et al. [2] for alternate proofs.)

The method adopted relies intensely on the precise asymptotic estimates for the heat kernel. In the similar way, H. Q. Li and his collaborator in [14, 15, 10], show that (1.1) and (1.2) hold in the Heisenberg type group $H(2n, m)$, see also [16] for the Grushin operators. For $SU(2)$ group, F. Baudoin and M. Bonnefont show that a modified form of (1.1) and (1.2) hold in [4]. The author himself shows that the gradient estimate (1.1) holds for the

three Brownian motion model in [22], see also [21] for the high dimensional Heisenberg group.

In this note, we shall focus on the nilpotent Lie group of rank two (It can also be called the n-Brownian motion model), which can be seen an another typical simpe example of hypoelliptic operator, but the structure is more complex than the Heisenberg (type) groups. Up to the author's knowledge, the method of H.Q. Li, [10]-[16], fails to study the precise gradient bounds in this context.

As the three Brownian motion model [22], we shall first look at the symmetries, that is we shall characterize all the vector fields which commute with the sub-Laplace Δ, see Proposition 3.1. The infinitesimal rotations are those vector fields which vanish at the orgin o and a radial function is a function which vanishes on infinitesimal rotations. In this case, although the Ricci curvature is everywhere $-\infty$, refer to [11, 2], we shall prove that the Γ_2 curvature is positive along the radial directions, as it is the case for the Heisenberg group and three Brownian motions model, see Theorem 4.5. The difficulty for general $n (n > 3)$ is that it is not easy to prove the positive curvature property directly even in the case of 4 Brownians motion model, since it is not easy to get the explicit, well organized solutions to the linear equations as the ones in the Proposition 3.1 in [22]. Even it is getting more and more complex as n grows. Inspired by the work of T. Melcher, c.f. [20], we will firstly prove L^1 heat kernel inequality for radial functions (see definition 3.5), and hence the positive property of Bakry-Emery Γ_2 curvature holds along the radial directions. As a consequence, the same form of gradient estimate (1.1) holds by combining the method developed by F. Baudoin and M. Bonnefont in [4] with the method in [14]. It is worth recalling that in [3], D. Bakry et al. have obtained the Li-Yau type gradient estimates for the three dimensional model group by applying Γ_2-techniques which plays an essential role in the paper. In our setting, it is easy to see that this type of gradient estimate also holds.

2 Nilpotent Lie group of rank two–$n-$dimensional Brownian motion model $N_{n,2}$

Let us recall the definition of nilpotent Lie group of rank two, see [9, 24].

Definition 2.1. *A linear space $\mathfrak{g}$ is a nilpotent Lie group of rank two if $\mathfrak{g} = V_1 \oplus V_2$, where V_1, V_2 are vector subspace of $\mathfrak{g}$, satisfying $V_2 = V_1 \oplus V_1, [V_1, V_2] = 0$ and $[V_2, V_2] = 0$. We denote $\mathfrak{N}_{n,2}$ the nilpotent algebra with n generators, denote $N_{n,2}$ the simple connected Lie group of rank two with the algebra $\mathfrak{N}_{n,2}$.*

Suppose V_1 is spanned by $X_i, 1 \le i \le n$ and V_2 is generated by $Y_{ij} := [X_i, X_j], i < j$. In this case, we have $[X_k, Y_{ij}] = 0,\ 1 \le i < j \le n, 1 \le k \le n$. The nature sub-Laplace operator is defined by

$$\Delta = \sum_{i=1}^{n} X_i^2. \tag{2.1}$$

Under the certain exponential map on $N_{n,2}$, without loss of any generality, we can assume X_i, Y_{ik} has the following form, see Lemma 4.1 in [9],

$$\begin{cases} X_i &= \partial_i + \frac{1}{2}\left(\sum_{k<i} x_k \hat{\partial}_{ki} - \sum_{k>i} x_k \hat{\partial}_{ik}\right), \\ Y_{ik} &= \hat{\partial}_{ik}, \end{cases} \tag{2.2}$$

for $1 \leq i, k \leq n$, with the notation $\partial_i = \frac{\partial}{\partial x_i}$, $\hat{\partial}_{ik} = \frac{\partial}{\partial y_{ik}}$. The reason why we call it the n Brownian motions model is that $\frac{1}{2}\Delta$ is the infinitesimal generator of the Markov process $\left(\{B_i\}_{1\leq i\leq n}, \{\frac{1}{2}\int_0^t B_i dB_{i+1} - B_{i+1} dB_i\}_{1\leq i\leq n}\right)$, where $\{B_i\}_{1\leq i\leq n}$ are n real standard independent Brownian motions.

By convention, for all $t \geq 0$, denote $P_t := e^{t\Delta}$ the associated heat semigroup generated by the canonical sub-Laplacian Δ, p_t the heat kernel of P_t at the origin o with respect to the Lebesgue measure on $\mathbb{R}^{\frac{n(n+1)}{2}}$.

For any function f, g defined on $N_{n,2}$, the carré du champ operators are, see [1, 12],

$$\begin{aligned}\Gamma(f,g) &:= \frac{1}{2}(\Delta(fg) - f\Delta g - g\Delta f)\\ &= \sum_{i=1}^{n} X_i f X_i g,\end{aligned}$$

and

$$\begin{aligned}\Gamma_2(f,f) &:= \frac{1}{2}(\Delta\Gamma(f,f) - 2\Gamma(f,\Delta f))\\ &= \sum_{i,j}(X_iX_jf)^2 + 2\sum_{i<j} X_jfX_iY_{ij}f - X_ifX_jY_{ij}f.\end{aligned}$$

Here the mixed term $\sum_{i<j} X_jfX_iY_{ij}f - X_ifX_jY_{ij}f$ prevent the existence of any constant ρ such that the curvature dimensional condition $CD(\rho,\infty)$ holds, see [11]. Nevertheless, we have the following Driver-Melcher inequality, see [7, 19],

$$\Gamma(P_tf, P_tf) \leq CP_t\Gamma(f,f).$$

for some positive constant C. The constant C here can be expressed explicitly following the method in [2] by dilation equation. For the Bakry-Emery heat kernel inequality (1.2), the methods deeply rely on the precise estimate on the heat kernel p_t and its differentials (see [13, 2, 10]). Up to the author's knowledge, these precise estimates are not known for the model $N_{n,2}$, neither the heat kernl inequality (1.2) (or so-called H. Q. Li inequality). Nevertheless, we shall prove that one of the key gradient estimates (1.1) holds, which would be a first step for the proof of the H. Q. Li inequality in this context. We remark that it does hold for radial functions, see Proposition 4.3, see also [20] for some other function classes.

For the heat kernel p_t, we have the following property, for $\vec{x} \in \mathbb{R}^n, \vec{y} \in \mathbb{R}^{\frac{n(n-1)}{2}}$, see [9],

$$p_t(\vec{x},\vec{y}) = t^{-n^2/2} p_1(\vec{x}/\sqrt{t}, \vec{y}/t), \tag{2.3}$$

hence it is enough to study the heat kernel p_t at time $t = 1$. For $t = 1$, we have for $\vec{x} = (x_1, \cdots, x_n)^t \in \mathbb{R}^n$, $\vec{y} = (y_{12}, \cdots, y_{n-1,n})^t \in \mathbb{R}^{\frac{n(n-1)}{2}}$, see P. 125, Theorem 1 in [9],

$$p(\vec{x},\vec{y}) := p_1(\vec{x},\vec{y}) = (2\pi)^{-\frac{n(n+2)}{2}} \int_{\mathbb{R}^{\frac{n(n-1)}{2}}} \exp\left(-i\sum_{k<l}\alpha_{kl}y_{kl}\right) \prod_{j=1}^{[\frac{n}{2}]} \varphi_j(A,\vec{x}) \prod_{k<l} d\alpha_{kl}, \tag{2.4}$$

where

$$\varphi_j(A,\vec{x}) = \frac{P_{2j-1}}{2}\left(\sinh\frac{P_{2j-1}}{2}\right)^{-1} \exp\left(-\frac{(\Omega^t\vec{x})^2_{2j-1} + (\Omega^t\vec{x})^2_{2j}}{2}\frac{P_{2j-1}}{2}\coth\frac{P_{2j-1}}{2}\right),$$

with A is an antisymmetric matrix with the entries $\{\alpha_{kl}\}_{k<l}$ in the upper triangular and Ω is the orthogonal matrix satisfying $\Omega^t A\Omega = P$, where P is a antisymmetric matrix formed by diagonal block of

$$\begin{pmatrix} 0 & P_{2k-1} \\ -P_{2k-1} & 0 \end{pmatrix}, \quad 1 \leq k \leq \frac{n}{2}, \text{ if n is even,}$$

and if k is odd, $1 \leq k \leq [\frac{n}{2}]$, the last block is 1×1 zero matrix, where iP_{2j-1} ($P_{2j-1} \in \mathbb{R}^+$) is the eigenvalue of the antisymmetric matrix A. Without loss of any generality, we can assume $P_1 \geq P_3 \geq \cdots \geq P_{2[\frac{n}{2}]} - 1$.

The natural distance, induced by the sub-Laplace Δ, is the Carnot-Carathéodory distance d. As usual, it can be defined from the gradient operator Γ only by, see [1, 24],

$$d(g_1, g_2) := \sup_{\{f:\Gamma(f)\leq 1\}} f(g_1) - f(g_2). \tag{2.5}$$

For this distance, we have the invariant and scaling properties, see [9, 24].

$$d(g_1, g_2) = d(g_2^{-1} \circ g_1, o) := d(g_2^{-1} \circ g_1), \text{ and } d(\gamma\vec{x}, \gamma^2\vec{y}) = \gamma d(\vec{x}, \vec{y}),$$

for all $g_1, g_2 \in N_{n,2}$, $\gamma \in \mathbb{R}^+$ and $\vec{x} \in \mathbb{R}^n, \vec{y} \in \mathbb{R}^{\frac{n(n-1)}{2}}$.

3 Radial functions

In this section, we will give the precise definition of radial functions. To this end, we study the rotation vectors in $N_{n,2}$.

Denote $\mathcal{T}$ be the linear space for such vectors (spanned by the vectors X_i, Y_j, $1 \leq i, jlen$), which commute to the sub-Laplace Δ. Now we trivially know that for $i < k$, Y_{ik} commutes to Δ since Y_{ik} commutes to X_i. Actually, there are lots of vectors who share this property. For simplicities, for $1 \leq i < j \leq n$,, denote

$$\theta_{ij} = x_j\partial_i - x_i\partial_j + \sum_{1\leq k<i} y_{kj}\hat{\partial}_{ki} - y_{ki}\hat{\partial}_{kj} + \sum_{i<k<j} y_{ik}\hat{\partial}_{kj} - y_{kj}\hat{\partial}_{ik} + \sum_{j<k\leq n} y_{jk}\hat{\partial}_{ik} - y_{ik}\hat{\partial}_{jk}, \tag{3.1}$$

and

$$\hat{X}_i = \partial_i - \frac{1}{2}\left(\sum_{k<i} x_k\hat{\partial}_{ki} - \sum_{k>i} x_k\hat{\partial}_{ik}\right).$$

In fact, $\{X_i\}_{1\leq i\leq n}$ ($\{\hat{X}_i\}_{1\leq i\leq n}$) can be called the left (right) invariant vectors respectively. For the vectors θ_{ij}, we have the following Lie relations: for $1 \leq i < j < k \leq n$,

$$[\theta_{ij}, \theta_{ik}] = \theta_{jk}. \tag{3.2}$$

Let us state the main result in this section.

Proposition 3.1.

$$\mathcal{T} = Linear\{\hat{X}_i, Y_i, \theta_{ij}, 1 \leq i, j \leq n\}.$$

Here Linear means the linear combination of vectors, with the constant coefficients.

Remark 3.2. *In particularly, for $n = 2$, this case is called the Heisenberg group, we can actually induce a group act such that X_i are corresponding to the left vector fields, see [9, 6, 2]. In this case we have $dim\mathcal{T} = 4$, and $\mathcal{T} = Span\{\hat{X}_1, \hat{X}_2, Y, \theta\}$, where*

$$\hat{X}_1 = \partial_{x_1} + \frac{x_2}{2}\partial_y,\ \hat{X}_2 = \partial_{x_2} - \frac{x_1}{2}\partial_y,\ \theta = x_1\partial_{x_2} - x_2\partial_{x_1}.$$

Here "Span" means the linear combination with the constant functions.

For $n = 3$. Actually, with changing the sign, we can also introduce a group act such that X_i are corresponding the left vector fields. Explicitly, see [9]

$$\begin{aligned} X_1 &= \partial_1 - \frac{x_2}{2}Y_3 + \frac{x_3}{2}Y_2, & \hat{X}_1 &= \partial_1 + \frac{x_2}{2}Y_3 - \frac{x_3}{2}Y_2; \\ X_2 &= \partial_2 - \frac{x_3}{2}Y_1 + \frac{x_1}{2}Y_3, & \hat{X}_1 &= \partial_1 + \frac{x_2}{2}Y_3 - \frac{x_3}{2}Y_2; \\ X_3 &= \partial_3 - \frac{x_1}{2}Y_2 + \frac{x_2}{2}Y_1, & \hat{X}_3 &= \partial_3 + \frac{x_1}{2}Y_2 - \frac{x_2}{2}Y_1, \end{aligned}$$

where $\hat{X}_i$ are the right vector fields, $Y_i = \partial_{y_i} := \hat{\partial}_i$. In this case, we have

$$\mathcal{T} = Linear\{\hat{X}_i, Y_i, \theta_i, 1 \le i \le 3\},$$

where

$$\begin{aligned} \theta_1 &= x_2\partial_3 - x_3\partial_2 + y_2\hat{\partial}_3 - y_3\hat{\partial}_2, \\ \theta_2 &= x_3\partial_1 - x_1\partial_3 + y_3\hat{\partial}_1 - y_1\hat{\partial}_3, \\ \theta_3 &= x_1\partial_2 - x_2\partial_1 + y_1\hat{\partial}_2 - y_2\hat{\partial}_1. \end{aligned}$$

It has been shown in [22].

To proof this Proposition, suppose any vector $X = \sum_i a_i X_i + \sum_{i<j} b_{ij} Y_{ij}$, satisfying $[\Delta, X] = 0$, where a_i, b_{ij} are the functions in $\{x_\cdot, y_{\cdot\cdot}\}$. Denote $W_{ij} = X_i X_j + X_j X_i$, then $X_i X_j = \frac{1}{2}(W_{ij} + Y_{ij})$. Note that

$$\begin{aligned} [\Delta, X] = &\sum_{i,j} X_i^2 a_j X_j + 2X_i a_j X_i X_j + 2a_j X_i Y_{ij} \\ &+ \sum_{i<j,k} X_k^2 b_{ij} Y_{ij} + 2X_k b_{ij} X_k Y_{ij} \\ = &\sum_{i,j} X_i^2 a_j X_j + \sum_{i<j} (X_i a_j + X_j a_i) W_{ij} + 2X_i a_i X_i^2 + \sum_{i<j} (X_i a_j - X_j a_i + \sum_k X_k^2 b_{ij}) Y_{ij} \\ &+ 2\sum_{i<j} (a_j X_i Y_{ij} - a_i X_j Y_{ij}) + \sum_{k \ne i,j, i<j} 2X_k b_{ij} X_k Y_{ij} + 2\sum_{i<j} X_i b_{ij} X_i Y_{ij} + 2\sum_{i<j} X_j b_{ij} X_j Y_{ij} \end{aligned}$$

thus we have

$$\sum_i X_i^2 a_j = 0, \tag{3.3}$$

$$X_i a_j = -X_j a_i, \qquad X_i a_i = 0, \tag{3.4}$$

$$\sum_k X_k^2 b_{ij} = 2X_j a_i,\ i < j, \qquad X_k b_{ij} = 0, k \ne i, j,\ i < j, \tag{3.5}$$

$$X_i b_{ij} = -a_j,\ i < j, \qquad X_j b_{ij} = a_i, i < j. \tag{3.6}$$

Lemma 3.3. *$a_i, 1 \leq i \leq n$ are linear functions in $\{x_i, 1 \leq i \leq n\}$, they are independent on $\{y_{ik}, i < k\}$.*

Proof. **Step1:** For fixed i, $j > i$, $[X_i, Y_{ij}] = 0$, Combining (3.5), we have for $i \neq k, l, k < l$, $X_i Y_{ij} b_{kl} = Y_{ij} X_i b_{kl} = 0$. Since $Y_{ij} = [X_i, X_j]$, again using the fact (3.5), we have $X_i^2 X_j b_{kl} = 0$, $i \neq k, l, k < l, i < j$. By choosing $l = j$, and using the fact (3.6), we have $X_i^2 a_k = 0, k < j, k \neq i$. In the same way, we have $X_i^2 a_l = 0$, $i < j < l$. Combining (3.4), we have

$$X_i^2 a_j = 0, \text{ for } 1 \leq i, j \leq n. \tag{3.7}$$

Step2: Again for $i < j, l$, $[X_i, Y_{ij}] b_{il} = 0$. By the fact that $[X_i, X_j] = Y_{ij}$ and (3.6), we have $X_i^2 X_j b_{il} + X_i X_j a_l = X_j X_i a_l - X_i X_j a_l$. Again we use the fact that $X_i^2 X_j b_{il} = 0$, which has been proved above, we get

$$2X_i X_j a_l = X_j X_i a_l, \text{ for } i < j, l.$$

For $i < j < l$, start from the fact that $[X_j, Y_{ij}] b_{jl}$=0, we have

$$2X_j X_i a_l = X_i X_j a_l, \ i < j < l.$$

Combining the above two equations, and (3.7) we have

$$X_i X_j a_l = X_j X_i a_l = 0, \ i \leq j \leq l.$$

Using $X_i a_j = -X_j a_i$, we have

$$X_i X_j a_l = 0, \ 1 \leq i, j, l \leq n. \tag{3.8}$$

Step3: By the fact that $Y_{ij} = [X_i, X_j]$, with (3.8), we have $Y_{ij} a_l = 0$, for $1 \leq l \leq n$, $i < j$. Thus $\{a_k\}_{1 \leq k \leq n}$ is independent on $\{y_{ik}\}_{1 \leq i < k \leq n}$, that is a_k is the function in $\{x_i\}_{1 \leq i \leq n}$. From the definition of X_k, we have $X_k a_j = \partial_k a_j$. By (3.8), i.e. for $1 \leq i, k \leq n$, $\partial_i^2 a_k = 0$, thus we can conclude a_k in linear function in x_i.

□

Thus we can give the explicit expression for a_i, for $1 \leq i \leq n$,

$$a_i = \sum_{j=1}^{n} A_{ij} x_j + B_i, \tag{3.9}$$

where A_{ij}, B_i are constants and A_{ij} satisfies $A_{ij} = -A_{ji}$.

Note that we can write

$$X = \sum_{i=1}^{n} a_i \partial_i + \sum_{i<j} c_{ij} \hat{\partial}_{ij}, \tag{3.10}$$

with $c_{ij} = b_{ij} + \frac{1}{2}(a_j x_i - a_i x_j)$. We have the following Lemma

Lemma 3.4. *For $1 \leq i, j \leq n$, c_{ij} are linear functions in $\{x_., y_{..}\}$.*

Proof. With the relation between b_{ij} with c_{ij} and the fact $X_i a_i = 0$, we have For $i < j$,

$$\begin{aligned} X_i b_{ij} = -a_j &\Longleftrightarrow \frac{1}{2} a_j = \frac{1}{2} x_i X_i a_j - X_i c_{ij}, \\ X_j b_{ij} = a_i &\Longleftrightarrow \frac{1}{2} a_i = \frac{1}{2} x_j X_j a_i + X_j c_{ij}, \\ X_k b_{ij} = 0 &\Longleftrightarrow X_k c_{ij} = \frac{1}{2}(x_i X_k a_j - x_j X_k a_i), \ k \neq i, j. \end{aligned} \tag{3.11}$$

Using $[X_i, X_j] = Y_{ij}$, the expression (3.9) and (3.11), through computation, we have, for $1 \leq i < j \leq n, \ 1 \leq k < l \leq n$,

$$Y_{ij} c_{kl} = \begin{cases} A_{il}, & 1 \leq i < j = k < l \leq n; \\ A_{jk}, & 1 \leq k < i = l < j \leq n; \\ A_{ki}, & 1 \leq i, k < j = l \leq n; \\ A_{lj}, & 1 \leq i = k < j, l \leq n; \\ 0, & \text{others}. \end{cases} \tag{3.12}$$

Combining (3.11) with (3.12) and the definition of X_i , through computation we have,

$$\partial_i c_{kl} = \begin{cases} -\frac{1}{2} B_l, & i = k; \\ \frac{1}{2} B_k, & i = l; \\ 0, & \text{others}. \end{cases} \tag{3.13}$$

(3.12) and (3.13) yield that for $1 \leq i, j \leq n$, $\deg c_{ij} \leq 1$ and we have the explicit expression for c_{kl}, for $k < l$,

$$c_{kl} = -\frac{1}{2} B_l x_k + \frac{1}{2} B_k x_l + \sum_{1 \leq i < k} A_{il} y_{ik} + \sum_{l < j \leq n} A_{jk} y_{lj} + \sum_{1 \leq i < l} A_{ki} y_{il} + \sum_{k < j \leq n} A_{lj} y_{kj} + D_{kl}, \tag{3.14}$$

where D_{kl} are constants. Combining (3.9), for $1 \leq i < j \leq n$, choose $A_{ij} = -A_{ji} = 1$ (respectively $D_{ij} = 1$), the other constants 0, we have $X = \theta_{ij}$ (respectively Y_{ij}). And for $1 \leq i \leq n$, choosing $B_i = 1$ and the other constants 0, we have $X = \hat{X}_i$.

we complete the proof.

□

Proof of Proposition 3.1. By the above Lemmas, we easily complete the proof. In the concrete case of $n = 2, 3$, from the equations (3.9) and (3.14), we can easily conclude.

□

Definition 3.5. *A C^2 function $f \colon N_{n,2} \to \mathbb{R}$, is called radial if it satisfies $\theta_{ij} f = 0$, for all $1 \leq i < j \leq n$.*

Remark 3.6. *(i). By the Lie relations (3.2), f is radial if and only if $\theta_{1k} f = 0$, for all $1 < k \leq n$. Clearly, constant functions are radial.*

(ii). If both f, g are radial, so do $k_1 f \pm k_2 g$, $f \cdot g$, for $k_1, k_2 \in \mathbb{R}$.

(iii). In particular the heat kernel $(p_t)_{t\geq 0}$ is radial. The reason is that for any function f, $1 < k \leq n$, $\theta_{1k}f(0) = 0$ and $\{\theta_{1k}\}_{1<k\leq n}$ commute with Δ, whence they commute with the semigroup $P_t = e^{t\Delta}$. Hence, for any function f, one has $P_t\theta_{1k}f = 0$, which, taking the adjoint of θ_{1k} under the Lebesgue measure, which is $-\theta_{1k}$, shows that for the density p_t of the heat kernel at the origin o, one has $\theta_{1k}p_t = 0$. This explains why any information about the radial functions in turns give information on the heat kernel itself.

(iv). In the Ph.D thesis of T. Melcher [20], the radial definition f is defined by $f(\vec{x}, \vec{y}..) = g(|\vec{x}|, \vec{y}..)$ for some smooth enough g. One defect in this definition is that the heat kernel p_t is not radial. To some extent, our definition of radial functions is more reasonable.

4 Γ_2 curvature

In this section, we will prove the associated Γ_2 curvature is positive on $N_{n,2}$. It generalizes the same property for the three Brownian motion model $N_{3,2}$ (c.f. [22], Proposition 3.1.). Up to the author's knowledge, the method adopted in [22] is not adapted easily in our setting, since it is not easy to express the solutions to the associated $\frac{n(n-1)}{2}$ equations regularly, even in the case of $n = 4$. To say nothing of proving the nonnegative property of Γ_2 curvature. Either it is hard to find out the certain parameter variables on which the radial functions depend for the case of $n > 3$ (We remark here that in the case of $n = 3$, radial functions depend on the norm of $\vec{x}$, $\vec{y}$, and their intersection angle $\langle\vec{x}, \vec{y}\rangle$). Inspired by the ad hoc methods adopted in Section 2.9.2 in the thesis of T. Melcher, c.f. [20], we will prove L^1 heat kernel inequality for radial functions f, and hence the nonnegative property for Γ_2 curvature holds along the radial directions.

For simplification, denote the following two gradient operators

$$\nabla f := (X_1 f, X_2 f, \cdots, X_n f), \hat{\nabla} f := (\hat{X}_1 f, \hat{X}_2 f, \cdots, \hat{X}_n f).$$

Let us first to the following key Lemma.

Lemma 4.1. *For any radial function f, we have*

$$\sum_{i=1}^{n}(X_i f)^2 = \sum_{i=1}^{n}(\hat{X}_i f)^2. \tag{4.1}$$

Proof. Recall that

$$X_i f = \partial_i f + \frac{1}{2}\left(\sum_{k<i} x_k \hat{\partial}_{ki} f - \sum_{k>i} x_k \hat{\partial}_{ik} f\right),$$

and

$$\hat{X}_i f = \partial_i f - \frac{1}{2}\left(\sum_{k<i} x_k \hat{\partial}_{ki} f - \sum_{k>i} x_k \hat{\partial}_{ik} f\right).$$

Hence

$$\sum_{i=1}^{n}(X_i f)^2 = \star + \sum_{i=1}^{n} \partial_i f\left(\sum_{k<i} x_k \hat{\partial}_{ki} f - \sum_{k>i} x_k \hat{\partial}_{ik} f\right),$$

and

$$\sum_{i=1}^{n}(\hat{X}_i f)^2 = \star - \sum_{i=1}^{n} \partial_i f \left(\sum_{k<i} x_k \hat{\partial}_{ki} f - \sum_{k>i} x_k \hat{\partial}_{ik} f \right),$$

where $\star$ is sum of square of $\partial_i f$ and $\frac{1}{2}\left(\sum_{k<i} x_k \hat{\partial}_{ki} f - \sum_{k>i} x_k \hat{\partial}_{ik} f\right)$. Thus to proof the desired result, we only need to prove

$$I := \sum_{i=1}^{n} \partial_i f \left(\sum_{k<i} x_k \hat{\partial}_{ki} f - \sum_{k>i} x_k \hat{\partial}_{ik} f \right) = 0. \tag{4.2}$$

Notice that

$$\begin{aligned}
I &= \sum_{i=1}^{n} \sum_{k=1}^{i-1} x_k \partial_i f \hat{\partial}_{ki} f - \sum_{i=1}^{n} \sum_{k=i+1}^{n} x_k \partial_i f \hat{\partial}_{ik} f \\
&= \sum_{k=1}^{n} \sum_{i=k+1}^{n} x_k \partial_i f \hat{\partial}_{ki} f - \sum_{i=1}^{n} \sum_{k=i+1}^{n} x_k \partial_i f \hat{\partial}_{ik} f \\
&\overset{(1)}{=} \sum_{i=1}^{n} \sum_{k=i+1}^{n} x_i \partial_k f \hat{\partial}_{ik} f - \sum_{i=1}^{n} \sum_{k=i+1}^{n} x_k \partial_i f \hat{\partial}_{ik} f \\
&= \sum_{i=1}^{n} \sum_{k=i+1}^{n} \left(x_i \partial_k f - x_k \partial_i f \right) \hat{\partial}_{ik} f,
\end{aligned}$$

where equality (1) follows from the exchange between i and k in the first term. Since f is radial ($\theta_{ij} f = 0$), by rotation vectors θ_{ij} defined in (3.1), we have

$$\begin{aligned}
I &= \sum_{i=1}^{n} \sum_{j=i+1}^{n} \left(x_i \partial_j f - x_j \partial_i f \right) \hat{\partial}_{ij} f \\
&= \sum_{i=1}^{n} \sum_{j=i+1}^{n} \hat{\partial}_{ij} f \Bigg(\sum_{k=1}^{i-1} (y_{kj} \hat{\partial}_{ki} f - y_{ki} \hat{\partial}_{kj} f) + \sum_{k=i+1}^{j-1} (y_{ik} \hat{\partial}_{kj} f - y_{kj} \hat{\partial}_{ik} f) \\
&\qquad\qquad + \sum_{k=j+1}^{n} (y_{jk} \hat{\partial}_{ik} f - y_{ik} \hat{\partial}_{jk} f) \Bigg) \\
&= \sum_{i=1}^{n} \sum_{j=i+1}^{n} \sum_{k=1}^{i-1} y_{kj} \hat{\partial}_{ki} f \hat{\partial}_{ij} f - \sum_{i=1}^{n} \sum_{j=i+1}^{n} \sum_{k=1}^{i-1} y_{ki} \hat{\partial}_{kj} f \hat{\partial}_{ij} f \\
&+ \sum_{i=1}^{n} \sum_{j=i+1}^{n} \sum_{k=i+1}^{j-1} y_{ik} \hat{\partial}_{kj} f \hat{\partial}_{ij} f - \sum_{i=1}^{n} \sum_{j=i+1}^{n} \sum_{k=i+1}^{j-1} y_{kj} \hat{\partial}_{ik} f \hat{\partial}_{ij} f \\
&+ \sum_{i=1}^{n} \sum_{j=i+1}^{n} \sum_{k=j+1}^{n} y_{jk} \hat{\partial}_{ik} f \hat{\partial}_{ij} f - \sum_{i=1}^{n} \sum_{j=i+1}^{n} \sum_{k=j+1}^{n} y_{ik} \hat{\partial}_{jk} f \hat{\partial}_{ij} f \\
&:= I_1 + I_2 + I_3 + I_4 + I_5 + I_6.
\end{aligned}$$

By change the order of summation, we have

$$I_1 = -I_6, I_2 = -I_3, I_4 = -I_5.$$

Thus we finish the proof. □

Remark 4.2. *The relation (4.1) holds for any functions, which satisfies (4.2). We would like to recommend the readers to the Proposition 2.28 in T. Melcher 's Ph. D thesis [20] for other class of functions satisfying (4.1).*

Now let us statement the L^1 heat kernel inequality for the radial functions, where the right invariant vector fields play an essential role.

Proposition 4.3. *For any radial function $f \in C_c^\infty(N_{n,2})$, we have, for any $t \geq 0$,*

$$|\nabla P_t f| \leq P_t(|\nabla f|).$$

Proof. Recall that for any function h, at the origin $o \in N_{n,2}$, we have $\nabla h = \hat{\nabla} h$. It follow, for radial function $f \in C_c^\infty(N_{n,2})$,

$$\begin{aligned} |\nabla P_t f|(o) &= |P_t \hat{\nabla} f|(o) \\ &\leq P_t(|\hat{\nabla} f|)(o) \\ &= P_t(|\nabla f|)(o), \end{aligned}$$

where the last equality follows from the Lemma 4.1. Thus

$$|\nabla P_t f|(g) \leq P_t(|\nabla f|)(g)$$

holds for any $g \in N_{n,2}$ by translation invariance. □

Remark 4.4. *The above Proposition can be compared with the Proposition 2.28 in [20].*

As a consequence, we have the following

Theorem 4.5. *For any compactly supported smooth, radial function f, for any $t \geq 0$, $g \in N_{n,2}$,*

(i) Positive curvature property. $\Gamma_2(f,f) \geq 0$.

(ii) LSI inequality. $P_t(f \log f)(g) - P_t(f) \log P_t(f)(g) \leq t P_t\left(\frac{\Gamma(f,f)}{f}\right)(g)$.

(iii) Isoperimetric inequality. $P_t(|f - P_t(f)(g)|)(g) \leq 4\sqrt{t} P_t(\Gamma(f)^{\frac{1}{2}})(g)$.

Proof. By Proposition 4.3, (i) follows from Proposition A, (ii) and (iii) follow from Theorem 6.1 and Theorem 6.2 in [2]. □

5 Gradient bounds for the heat kernels

As done in [3], we have the following Li-Yau type inequality holds.

Proposition 5.1. *There exist positive constants C_1, C_2, C_3 (dependent on n) such that for any positive function f, if $u = \log P_t f$, we have*

$$\partial_t u \geq C_1 \Gamma(u) + C_2 t \sum_{1 \leq i < j \leq n} |Y_{ij} u|^2 - \frac{C_3}{t}.$$

Proof. Since the proof closely follows [3], we skip the proof. We would like recommend the readers' to [3] and the interesting paper [5]. □

As a consequence, we have the following Harnack inequality: There exist positive constants A_1, A_2(dependent on n, see [5] for exact expression for A_1, A_2), for $t_2 > t_1 > 0$, and $g_1, g_2 \in N_{n,2}$,

$$\frac{p_{t_1}(g_1)}{p_{t_2}(g_2)} \leq \left(\frac{t_2}{t_1}\right)^{A_1} e^{A_2 \frac{d^2(g_1,g_2)}{t_2-t_1}}. \tag{5.1}$$

Let us state the first result of the gradient estimate for the heat kernel.

Proposition 5.2. *There exists a constant $C > 0$(dependent on n) such that for $t > 0$, $g = (\vec{x}, \vec{y}) \in N_{n,2}$,*

$$\sqrt{\Gamma(\log p_t)(g)} \leq C\left(\frac{d(g)}{t} + \frac{1}{\sqrt{t}}\right),$$

where $p_t(g)$ denotes the density of P_t at o and $d(g)$ denotes the Carnot-Carathéodory distance between o and g.

Proof. Following [4] as in [22], for $0 < s < t$, let $\Phi(s) = P_s\big(p_{t-s}\log p_{t-s}\big)$, we have

$$\Phi'(s) = P_s\big(p_{t-s}\Gamma(\log p_{t-s})\big),\ \Phi''(s) = 2P_s\big(p_{t-s}\Gamma_2(\log p_{t-s})\big).$$

By Theorem 4.5, Φ'' is positive, whence Φ' is non-desceasing, thus

$$\int_0^{\frac{t}{2}} \Phi'(s)ds \geq \frac{t}{2}\Phi'(0).$$

That is

$$p_t\Gamma(\log p_t) \leq \frac{2}{t}\big(P_{t/2}(p_{t/2}\log p_{t/2}) - p_t \log p_t\big).$$

The right hand side can be bounded by applying the above Harnack inequality (5.1) and the basic fact $p_{t/2}(g) \leq p_{t/2}(o)$, for all $g \in N_{n,2}$. We have

$$\sqrt{\Gamma(\log p_t)(g)} \leq C\left(\frac{d(g)}{t} + \frac{1}{\sqrt{t}}\right).$$

□

Proposition 5.3. *For $g = (\vec{x}, \vec{y}) \in N_{n,2}$ satisfying $d(g) \leq 1$, there exists a positive constant C (dependent on n), such that*

$$\sqrt{\Gamma(p)(g)} \leq Cd(g).$$

Proof. Recall that we have the precise expression of the heat kernel, see (2.4). To estimate $\Gamma(p)(g)$, denote the orthogonal matrix $\Omega = (\omega_{ij})_{1\leq i,j\leq n}$, which appear in the $\varphi_j(A, \vec{x})$ in (2.4), we have

$$|\varphi_j(A, \vec{x})| \leq \frac{P_{2j-1}}{2}(\sinh\frac{P_{2j-1}}{2})^{-1}$$

$$|\partial_i\varphi_j(A, \vec{x})| \leq (\frac{P_{2j-1}}{2})^2(\sinh\frac{P_{2j-1}}{2})^{-1}\coth\frac{P_{2j-1}}{2}(\omega_{i,2j-1}^2 + \omega_{i,2j}^2)|x_i|.$$

It yields, for $1 \le i \le n$,

$$\partial_i p(\vec{x},\vec{y}) \le (2\pi)^{-\frac{n(n+2)}{2}} \int_{\mathbb{R}^{\frac{n(n-1)}{2}}} \prod_{j=1}^{[\frac{n}{2}]} \frac{P_{2j-1}}{2} (\sinh \frac{P_{2j-1}}{2})^{-1} \sum_{j=1}^{[\frac{n}{2}]} \frac{P_{2j-1}}{2} \coth \frac{P_{2j-1}}{2} (\omega_{i,2j-1}^2 + \omega_{i,2j}^2)|x_i| \prod_{k<l} d\alpha_{kl}$$

$$\le (2\pi)^{-\frac{n(n+2)}{2}} |x_i| \int_{\mathbb{R}^{\frac{n(n-1)}{2}}} \frac{P_1}{2} \coth \frac{P_{2[\frac{n}{2}]-1}}{2} \prod_{j=1}^{[\frac{n}{2}]} \frac{P_{2j-1}}{2} (\sinh \frac{P_{2j-1}}{2})^{-1} \prod_{k<l} d\alpha_{kl}$$

where we use the fact that for $1 \le i \le n$, $\sum_{1 \le j \le 2[\frac{n}{2}]-1} \omega_{ij}^2 \le 1$, which is the consequence of the fact that Ω is orthogonal matrix. Also we have for $k' < l'$,

$$\hat{\partial}_{k'l'} p(\vec{x},\vec{y}) \le (2\pi)^{-\frac{n(n+2)}{2}} \int_{\mathbb{R}^{\frac{n(n-1)}{2}}} |\alpha_{k'l'}| \prod_{j=1}^{[\frac{n}{2}]} \frac{P_{2j-1}}{2} (\sinh \frac{P_{2j-1}}{2})^{-1} \prod_{k<l} d\alpha_{kl}.$$

It follows,

$$\sum_{i=1}^{n} |X_i p| \le (2\pi)^{-\frac{n(n+2)}{2}} |\vec{x}| (W_1 + W_2), \tag{5.2}$$

where

$$W_1 = \int_{\mathbb{R}^{\frac{n(n-1)}{2}}} \frac{P_1}{2} \coth \frac{P_{2[\frac{n}{2}]-1}}{2} \prod_{j=1}^{[\frac{n}{2}]} \frac{P_{2j-1}}{2} (\sinh \frac{P_{2j-1}}{2})^{-1} \prod_{k<l} d\alpha_{kl},$$

and

$$W_2 = \int_{\mathbb{R}^{\frac{n(n-1)}{2}}} \sum_{k<l} |\alpha_{kl'}| \prod_{j=1}^{[\frac{n}{2}]} \frac{P_{2j-1}}{2} (\sinh \frac{P_{2j-1}}{2})^{-1} \prod_{k<l} d\alpha_{kl}$$

with the restriction

$$\alpha := \sum_{k<l} \alpha_{kl}^2 = \sum_{j=1}^{[\frac{n}{2}]} P_{2j-1}^2,$$

which follows from the fact that both sides are the half of the trace of $-A^2 = A^t A$. Note that for $1 \le j \le [\frac{n}{2}]$, $P_{2j-1} \le \sqrt{\alpha}$ and $(\sinh x)^{-1} \le 4e^{-x}$ for $|x| \ge \frac{1}{2}$, we have for positive constants C_1, C_2,

$$W_1 \le C_1 vol(B_1(0)) + C_2 \int_{B_1^c(0)} \alpha^{\frac{[\frac{n}{2}]+1}{2}} e^{-\sqrt{\alpha}} \prod_{k<l} d\alpha_{kl}$$

which is obviously bounded. Similarly, we have W_2 is bounded.

Combining with (5.2) and the fact $|x| \le d(g) \le 1$ (see [24]), we have

$$\sqrt{\Gamma(p)(g)} \le C_1 |x| (W_1 + W_2) \le C_2 |x| \le C d(g).$$

$\square$

Here is an analogue result of Theorem B in the case $N_{n,2}$.

Proposition 5.4. *There exists a constant $C > 0$ (dependent on n) such that for $t > 0$, $g = (\vec{x},\vec{y}) \in N_{n,2}$,*

$$\sqrt{\Gamma(\log p_t)(g)} \le \frac{C d(g)}{t}.$$

Proof. Taking $t = 1$ in Proposition 5.2, we have

$$\sqrt{\Gamma(\log p)(g)} \leq C\,(d(g) + 1)\,.$$

If $d(g) \geq 1$, it is trivial to get the desired result from the above gradient estimate. For the case $d(g) \leq 1$, note that the heat kernel is bounded below by a positive constant (see [24]), combining with Proposition 5.3, we have for some positive constant C (dependent on n),

$$\Gamma(\log p)(g) \leq Cd(g), \quad g \in N_{n,2}.$$

The desired result follows by the time scaling property (2.3). □

Remark 5.5. *In a forthcoming paper, we shall study the gradient estimates for the heat kernels of the sub-elliptic operators, which satisfy the generalized curvature dimension inequalities $CD(\rho_1, \rho_2, k, d)$ introduced by F. Baudoin and N. Garofalo in [5].*

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